



8.11 Row operations

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Introduction

- In this topic, we will
 - Review the row operations on a matrix
 - See that each row operation is a linear operator, and thus can be represented by a square matrix
 - Find these matrices
 - Observe that these matrices are invertible





Row operations

- There are three row operations:
 - Swapping two rows
 - Adding a multiple of one row onto another
 - Multiplying a row by a non-zero scalar
- Previously, we represented these operations by an italicized capital R with a subscript

$$R_{R_i \leftrightarrow R_j} \quad R_{\alpha R_i \rightarrow R_j} \quad R_{\alpha R_i}$$

- Recall also that all linear maps are represented by such letters





Swapping two rows

- Suppose that $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$,
so A is representable by an $m \times n$ matrix
- Recall that the matrix-matrix multiplication BA can be found by multiplying each column vector of A by B , so
$$BA = (B\mathbf{a}_1 \cdots B\mathbf{a}_n)$$
- Question: How can we swap two entries of each $\mathbf{a}_k$
- Now, swapping two entries in a vector is a linear mapping,
for it doesn't matter if we swap those entries in $\beta \mathbf{u}_1 + \gamma \mathbf{u}_2$
or swap the two entries of $\mathbf{u}_1$ and $\mathbf{u}_2$ and then take the same
linear combination of the results
 - The resulting vector will be the same dimension: m
 - Thus, the linear operator is an $m \times m$ matrix
- Question: what is the matrix $R_{\mathbf{R} i \leftrightarrow \mathbf{R} j}$?





Swapping two rows

$$R_{Ri \leftrightarrow Rj} = \begin{matrix} & \begin{matrix} 1 & \dots & i & \dots & j & \dots & m \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ m \end{matrix} & \begin{pmatrix} \textcolor{yellow}{1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & \textcolor{yellow}{1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \boxed{0} & 0 & \dots & 0 & \boxed{0} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \textcolor{yellow}{1} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & \textcolor{yellow}{1} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \boxed{0} & 0 & \dots & 0 & \boxed{0} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \textcolor{yellow}{1} & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \textcolor{yellow}{1} \end{pmatrix} \end{matrix}$$





Swapping two rows

- Suppose we want to swap Rows 2 and 5 of this matrix
 - Multiply on the left by $R_{R2 \leftrightarrow R5}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 & 5 \\ 0 & 1 & 7 \\ 0 & 4 & 2 \\ 0 & 2 & 3 \\ 0 & -8 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 5 \\ 0 & -8 & 2 \\ 0 & 4 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & 7 \end{pmatrix}$$





Swapping two rows

- Now, we can undo the operation of swapping two rows by once again swapping two rows
 - Consequently, this linear operator is its own inverse:

$$R_{Ri \leftrightarrow Rj}^{-1} = R_{Ri \leftrightarrow Rj}$$

$$R_{Ri \leftrightarrow Rj} R_{Ri \leftrightarrow Rj} = I$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I_5$$





Adding a multiple of one row onto another

- Again, $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$ and is represented by an $m \times n$ matrix
- Again, we have

$$BA = (Ba_1 \cdots Ba_n)$$

- Question: How can we add a multiple of one row onto another?
- Now, adding a multiple of one entry onto another is linear,
for we can do this mapping after calculating $\beta \mathbf{u}_1 + \gamma \mathbf{u}_2$
or we can do this mapping first on both $\mathbf{u}_1$ and $\mathbf{u}_2$ and
then take the same linear combination of the results
 - The resulting vector will be the same dimension: m
 - Thus, the linear operator is an $m \times m$ matrix
- Question: what is the matrix $R_{\alpha Ri \rightarrow Rj}$?





Adding a multiple of one row onto another

$$R_{\alpha Ri \rightarrow Rj} = \begin{matrix} & \begin{matrix} 1 & \dots & i & \dots & j & \dots & m \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ \bar{i} \\ \vdots \\ \bar{j} \\ \vdots \\ m \end{matrix} & \left(\begin{array}{cccccccccc} \mathbf{1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & \mathbf{1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathbf{1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \mathbf{1} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & \mathbf{1} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \alpha & 0 & \dots & 0 & \mathbf{1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \mathbf{1} & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \mathbf{1} \end{array} \right) \end{matrix}$$





Adding a multiple of one row onto another

- Suppose we want to add 0.5 times Rows 2 onto Row 3:
 - Multiply on the left by $R_{0.5R2 \rightarrow R3}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 & 5 \\ 0 & -8 & 2 \\ 0 & 4 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 5 \\ 0 & -8 & 2 \\ 0 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 7 \end{pmatrix}$$





Swapping two rows

- Now, we can undo the operation of adding a multiple of one row onto another?
 - Yes, subtract that same multiple of the one row from the other:

$$R_{\alpha Ri \rightarrow Rj}^{-1} = R_{-\alpha Ri \rightarrow Rj}$$

$$R_{\alpha Ri \rightarrow Rj} R_{-\alpha Ri \rightarrow Rj} = R_{-\alpha Ri \rightarrow Rj} R_{\alpha Ri \rightarrow Rj} = I$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I_5$$





Multiplying a row by a non-zero scalar

- Again, $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$ and is represented by an $m \times n$ matrix
- Again, we have

$$BA = (Ba_1 \cdots Ba_n)$$

- Question: How can we multiply a row by a non-zero scalar?
- Now, multiplying an entry by a non-zero scalar is linear,
 - for we can do this mapping after calculating $\beta \mathbf{u}_1 + \gamma \mathbf{u}_2$
 - or we can do this mapping first on both $\mathbf{u}_1$ and $\mathbf{u}_2$ and
 - then take the same linear combination of the results
- The resulting vector will be the same dimension: m
- Thus, the linear operator is an $m \times m$ matrix
- Question: what is the matrix $R_{\alpha Ri}$?





Multiplying a row by a non-zero scalar

$$R_{\alpha Ri} = \begin{matrix} & \begin{matrix} 1 & \dots & i & \dots & j & \dots & m \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ m \end{matrix} & \left(\begin{array}{cccccccccccc} \mathbf{1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & \mathbf{1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \boxed{\alpha} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \mathbf{1} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & \mathbf{1} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & \mathbf{1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \mathbf{1} & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \mathbf{1} \end{array} \right) \end{matrix}$$





Multiplying a row by a non-zero scalar

- Suppose we want to multiply Rows 2 by -0.125 :
 - Multiply on the left by $R_{-0.125R2}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.125 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1.666 \\ 0 & -8 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 3.5 \\ 0 & 0 & 7.25 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1.666 \\ 0 & 1 & -0.25 \\ 0 & 0 & 3 \\ 0 & 0 & 3.5 \\ 0 & 0 & 7.25 \end{pmatrix}$$





Multiplying a row by a non-zero scalar

- Now, we can undo the operation of multiplying a row by a non-zero scalar?
 - Yes, multiply that row by the multiplicative inverse of that scalar:

$$R_{\alpha Ri}^{-1} = R_{\alpha^{-1} Ri}$$

$$R_{\alpha^{-1} Ri} R_{\alpha Ri} = R_{\alpha Ri} R_{\alpha^{-1} Ri} = I$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -0.125 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I_5$$





Summary

- Following this topic, you now
 - Know that each row operation can be represented by a matrix
 - Understand that each of these matrices has inverses





References

- [1] https://en.wikipedia.org/wiki/Elementary_matrix





Acknowledgments

None so far.





Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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